Problems from the American Mathematical Monthly, due May 31, 2024

12433. *Proposed by Etan Ossip, student, Queen's University, Kingston, ON, Canada.* For $x > 1$, prove

$$\frac{i}{2}\int_{-\infty}^{\infty} \frac{\tanh(\pi t)}{\left(\frac{1}{2}+it\right)^x}\,dt = \zeta(x),$$

where ζ is the Riemann zeta function.

12434. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Ploieşti, Romania.* Let $a_1, \dots, a_n$ be real numbers such that $a_1 \geq \cdots \geq a_n \geq 0$. Prove

$$\left(\frac{a_1a_2\cdots a_{n-1} + a_2a_3\cdots a_n + \cdots + a_na_1\cdots a_{n-2}}{n}\right)^2 \leq \left(\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n}\right)^{n-1}.$$

12435. *Proposed by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy.* For a positive integer n, let $d(n)$ be the number of positive divisors of n, let $\phi(n)$ be Euler's totient function (the number of integers in $\{1, \dots, n\}$ that are relatively prime to n), and let $q(n) = d(\phi(n))/d(n)$. Find $\inf_n q(n)$ and $\sup_n q(n)$.

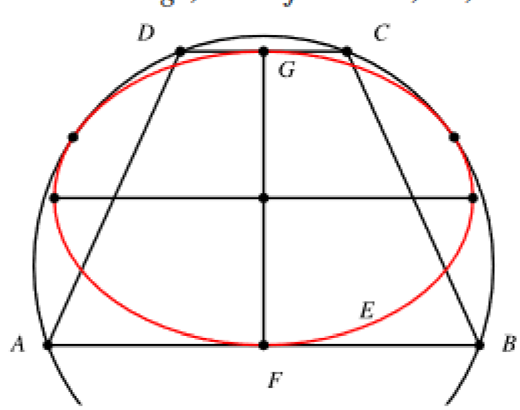
12436. *Proposed by Lorenzo Sauras-Altuzarra, Vienna University of Technology, Vienna, Austria.* For a positive integer n, evaluate

$$\prod_{k=1}^{n}\left(x + \sin^2\left(\frac{k\pi}{2n}\right)\right).$$

12437. *Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania, and Mihai Opincariu, Brad, Romania.* For any n-by-n complex matrix M, prove

$$\operatorname{rank}(M) + \operatorname{rank}\left(M - M^3\right) = \operatorname{rank}\left(M - M^2\right) + \operatorname{rank}\left(M + M^2\right).$$

12438. *Proposed by Robert Foote, Wabash College, Crawfordsville, IL, and Gregory Galperin, Eastern Illinois University, Charleston, IL.* Let $ABCD$ be an isosceles trapezoid inscribed in a circle γ with AB parallel to CD. Let F bisect AB and G bisect CD. Let E be the ellipse with minor axis FG and major axis of length AC. Prove that E is internally tangent to γ at two points.



12439. *Proposed by Haoran Chen, Xi'an Jiaotong–Liverpool University, Suzhou, China.* An n-by-n-by-n cube is formed from n^3 unit cubes. The removal of some of the unit cubes leaves a solid S such that

(1) the projection of S onto each face of the original cube is an n-by-n square; and

(2) from each unit cube in S one can reach any other unit cube in S along a chain of cubes each of which shares a face with its predecessor.

What is the minimum number of unit cubes that S can have?

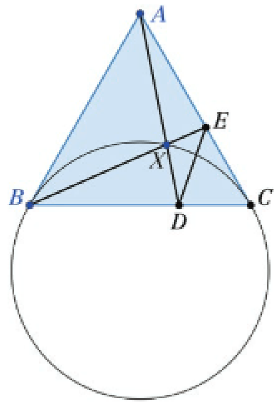
Problems from Math Horizons, due May 31, 2024

A Curious Identity on Triangular Areas (P462). Yagub Aliyev (ADA University) posed this problem. As depicted in figure 1, segments AB and AC are two tangents from point A to some circle such that the triangle ABC is equilateral. Let X be an arbitrary point on the smaller arc $\widehat{BC}$. Suppose that the segments AX and BX intersect BC and AC at the points D and E, respectively. Prove that

$$\sqrt{\frac{1}{[BDE]}+\frac{1}{[ADE]}}=\sqrt{\frac{1}{[ABD]}}+\sqrt{\frac{1}{[ACD]}}$$

where $[PQR]$ denotes the area of the triangle PQR.

Figure 1. An equilateral triangle derived from a circle.



Oppenheimer Numbers (P463). Brian J. Shelburne (Wittenberg University) asked this question. The book *American Prometheus*, which inspired the movie *Oppenheimer*, contains an intriguing story about J. Robert Oppenheimer and the number 3528. In short, Oppenheimer forgot the house number when searching for a New Year's party, only recalling interesting properties satisfied by the number. Fortunately, he eventually identified 3528 as the correct address.

Let $N = a_1 \dots a_n b_1 \dots b_n$ be a $2n$-digit number. Define N to be an *Oppenheimer number* if it satisfies the following conditions:

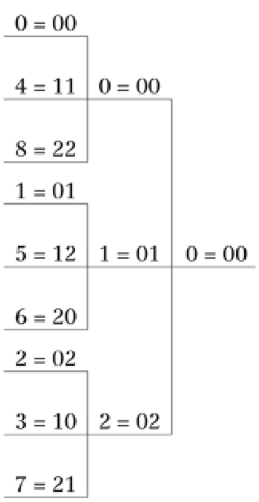
1) The two numbers created from the halves of N satisfy $\gcd(a_1 \dots a_n, b_1 \dots b_n) > 1$.
2) The digit a_1 is greater than 1 and divides N.
3) The digit b_n divides N.
4) The numbers a_1, $\gcd(a_1 \dots a_n, b_1 \dots b_n)$, and b_n are mutually coprime.

For example, 3528 is a four-digit Oppenheimer number. Is 3528 the only four-digit Oppenheimer number? Do $2n$-digit Oppenheimer numbers exist for all $n \geq 2$?

***Jeopardy*! Brackets (P464).** Stan Seltzer (Ithaca College) and Shai Simonson (Stonehill College) suggested this problem connected to their article "Bracket Challenge" (see page 20). The game show *Jeopardy*! features tournaments where each game consists of three contestants and one winner advancing. Define a seeded bracket with 3^k teams, using 0 to $3^k - 1$ as seeds, where three teams compete per game with one winner advancing to be *k-balanced* if the sum of the seeds is constant for each round when the top seed advances. For example, the nine-player bracket shown in figure 2 is 2-balanced.

A k-balanced bracket can be constructed using ternary (base 3) representations for the seeds as follows. Start with 0, 1, and 2 in the finals with 0 as the top seed. To generate the round with 3^2 teams, prepend a 0 to each (obtaining 00, 01, 02), and to each add 11 and 22 (base 3) with no carries. This gives the seeds expressed in ternary of the 2-balanced bracket in figure 2. Iterating produces a k-balanced bracket.

Figure 2. A 2-balanced bracket. The seeds are expressed using both decimal and (two-digit) ternary representations.



1) Using the 3-balanced bracket for 27 teams constructed as just described, what round would the 6 and 16 seeds meet?
2) What is the general algorithm to answer such a question for a k-balanced bracket constructed as just described?
3) There are other ways to construct k-balanced brackets. How many k-balanced brackets are there?

A Few Good Integer Polynomials (P465). Anthony Bevelacqua (University of Norh Dakota) proposed this problem. Let $p(x)$ be a polynomial with integer coefficients. Call $p(x)$ *good* if $p(x)$ takes the value 1 at exactly $r > 0$ distinct integers and $p(x)$ has exactly $s > 0$ distinct integer roots. For a good polynomial $p(x)$, show that (r, s) must be one of (1,1), (1,2), or (2,1).

Problems from Kappa Mu Epsilon's journal, *The Pentagon*, due May 31, 2024

Problem 920. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let α be the golden ratio. Show that

$$\sum_{i=0}^{\infty} \frac{i}{\alpha^i} \left(\sum_{j=0}^{\infty} \left[\lim_{n\to\infty} \frac{F_n^2 + F_{n+2}^2 - F_{n+1}F_{n+3}}{F_{n-1}F_{n+2}}\right]^j\right)^{-1} = \alpha$$

Problem 921. *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve in real numbers the following equation:

$$\log_2(x^2 + 2^x) + (x^2 - 1) * 2^{x+1} + x^4 + x^2 + 2^x = 3 * 4^x + x + 1$$

Problem 922. *Proposed by Toyesh Prakash Sharma,*

Let F_n be the nth Fibonacci number defined by $F_1 = 1, F_2 = 1$ and for all n≥3,$F_n = F_{n-1} + F_{n-2}$. Prove that $\sum_{n=1}^{\infty} (\frac{1}{9})^{F_{n+2}}$ is an irrational number but not a transcendental number.

Problem 923. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let n≥1 be an integer. Compute

$$\lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n*1}} \sum_{k=0}^{\infty} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}$$

Problem 924. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let a, b, c be three positive real numbers. Prove that

$$\frac{a}{4b+7\sqrt{ab}} + \frac{b}{4c+7\sqrt{bc}} + \frac{c}{4a+7\sqrt{ca}} \geq \frac{3}{11}$$

Problem 925. *Proposed Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Prove that in any triangle ABC with usual notations (R = circumradius, r = inradius, s=semiperimeter, m_a = median from vertex A) the following inequality is true:

$$2\sum m_a \leq 3\sqrt{\frac{R(s^2+r^2+Rr)}{2r}}$$

Problem 926. *Proposed by the editor.*
Prove that the sequence
$a_1 = 1, a_2 = 1, a_n = a_{n-1} + 2 * a_{n-2}$ for all $n > 2$
gives the number of integers between 2^n and 2^{n+1} which are divisible by 3.

Problem 927. *Proposed by the editor.*

Find the area below the two lines 8x+5y= 976 and 6x+5y = 792 that lies in the first quadrant.

Problems from Crux MathemAttic, due June 15, 2024

MA266. *Proposed by Elton Papanikolla.*

Let $ABCD$ and $WXYZ$ be two squares that share the same center such that $WX || AB$ and $|WX| < |AB|$. Lines CX and AB intersect at P and lines CZ and AD intersect at Q. If points P, W and Q are collinear, compute the ratio $|AB| : |WX|$.

MA267. *Proposed by Toyesh Prakash Sharma.*

Consider $0 \leq x, y, z \leq \frac{\pi}{2}$ and $x + y + z = \frac{\pi}{2}$. Show that

$$\cos x + \cos y + \cos z \geq 1 + \sin x + \sin y + \sin z.$$

MA268. A local high school math club has 12 students in it. Each week, 6 of the students go on a field trip. If each pair of students have been on at least one field trip together, determine the minimum number of field trips that could have happened.

MA269. Alice and Bob play a game, taking turns, playing in a row of n seats. On a player's turn, he or she places a coin on any seat, provided there is no coin on that seat or on an adjacent seat. Alice moves first. The player who does not have a valid move loses the game.

a) Show that Alice has a winning strategy when $n = 6$.

b) Show that Bob has a winning strategy when $n = 8$.

MA270. Consider a convex quadrilateral $ABCD$. Let rays BA and CD intersect at E, rays DA and CB intersect at F, and the diagonals AC and BD intersect at G. It is given that the triangles DBF and DBE have the same area. Given that the area of triangle ABD is 4 and the area of triangle CBD is 6, compute the area of triangle EFG.

Problems from Crux Olympiad Corner, due June 15, 2024

OC676. There are at least two positive integers along a circle. For any two neighbouring integers one is either twice as big as the other or five times as big as the other. Can the sum of all these integers be 2023?

OC677. A region of a chessboard is defined as a number of squares joined edge to edge and not enclosing any other square. An 8×8 board is divided into eight regions each consisting of eight squares. The entire 8×8 board is covered by points. Without knowing the actual partition, is it possible to choose nine interior points of the board so that at most two points are in the interior of each region?

OC678. The chord DE of the circumcircle of triangle ABC intersects AB and BC at P and Q respectively, with P between D and Q. F and G are points on AP and CQ respectively, such that $\angle ADF = \angle FDP$ and $\angle CEG = \angle GEQ$. Prove that if $DFGE$ is a cyclic quadrilateral, then $CAPQ$ is also a cyclic quadrilateral.

OC679. Let n be any positive integer. Prove that

$$\sum_{i=1}^{n} \frac{1}{(i^2+i)^{3/4}} > 2 - \frac{2}{\sqrt{n+1}}.$$

OC680. Silverio writes a finite sequence of As and Gs on a nearby blackboard. He then performs the following operation: if he finds at least one occurrence of the string AG, he chooses one at random and replaces it with GAAA. He performs this operation repeatedly until there is no more AG string on the blackboard. Show that for any initial sequence of As and Gs, Silverio will eventually be unable to continue doing the operation.

Problems from Crux Mathematicorum, due June 15, 2024

4931. *Proposed by Michel Bataille.*

Let a, b be nonnegative real numbers such that $\lfloor\sqrt{b}\rfloor = \lfloor\sqrt{a}\rfloor + 1$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Prove that $\lfloor\sqrt{b+m}\rfloor = \lfloor\sqrt{a+m}\rfloor + 1$ for infinitely many positive integers m.

4932. *Proposed by George Apostolopoulos.*

On the sides AB and AC of a triangle ABC, consider the interior points E and D, respectively, such that

$$\left(\frac{AE}{EB}\right)^2 + \left(\frac{AD}{DC}\right)^2 = 1.$$

The segments BD and CE intersect at point P. Find the ratio of the areas of quadrilateral $EBCD$ and triangle PBC.

4933. *Proposed by Mihaela Berindeanu.*

For $a, b, c > 0$ show that $\displaystyle\frac{a^2}{b\sqrt{ab+5bc}} + \frac{b^2}{c\sqrt{bc+5ca}} + \frac{c^2}{a\sqrt{ca+5ab}} \geq \frac{\sqrt{6}}{2}$.

4934. *Proposed by Dong Luu.*

Let ABC be a triangle and BE, CF be its altitudes. The points M, N belong to the line BC such that $\overrightarrow{BM} = -\overrightarrow{CN}$ (M, N are different from B, C). Denote by I and J the center of (CNE) and (BMF), respectively. Prove that the lines BI, CJ and AH are either concurrent or parallel.

4935. *Proposed by Daniel Sitaru.*

Let $A, B \in M_3(\mathbb{C})$, $C, D \in M_5(\mathbb{C})$ and $E, F \in M_7(\mathbb{C})$. Show that

$|rank(AB) - rank(BA)| + |rank(CD) - rank(DC)| + |rank(EF) - rank(FE)| \leq 6.$

4936. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with centroid G, medians AD, BE, CF, and altitudes BM and CN. Prove that the midpoint of AD lies on the line MN if and only if A, F, G, E are concyclic.

4937. *Proposed by Michel Bataille.*

Let a be a positive real number and let $f : [0, \infty) \to \mathbb{R}$ be a continuous, a-periodic function. Prove that if $b > a$, then

$$\int_0^b \int_0^a \frac{f(x+y)}{x+y}\, dx dy = a \int_a^{a+b} \frac{f(t)}{t}\, dt + b \int_b^{a+b} \frac{f(t)}{t}\, dt.$$

4938. *Proposed by Ivan Hadinata.*

Let a, b, c be positive real numbers such that $3abc = ab + ac + bc$. Show that

$$\frac{a(a^4 + b^2c^2)}{b+c} + \frac{b(b^4 + a^2c^2)}{a+c} + \frac{c(c^4 + a^2b^2)}{a+b} \geq a + b + c.$$

4939. *Proposed by Nguyen Minh Ha.*

Prove that a triangle ABC for which $m_a = l_b = h_c$ is equilateral, where m_a is the median from the vertex A, l_b is the bisector of $\angle CBA$, and h_c is the altitude from vertex C.

4940. *Proposed by Paul Bracken.*

Prove that for $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \binom{2k}{k}^2 \frac{k}{2^{4k}(2k-1)^2} = \frac{n(n+1)^2}{2^{4n+2}(2n+1)^2} \binom{2n+2}{n+1}^2.$$

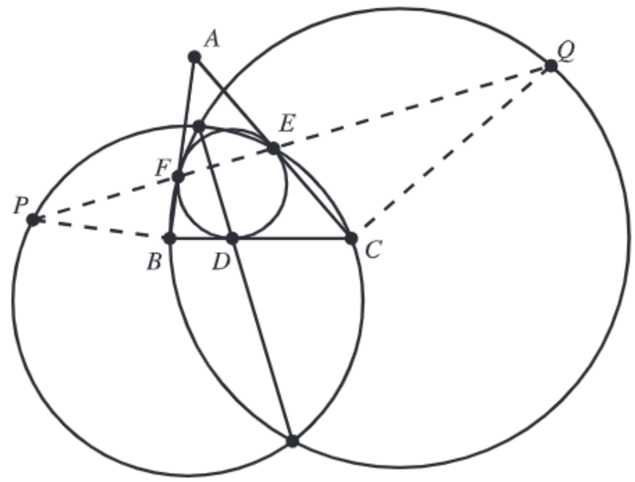
Problems from the American Mathematical Monthly, due June 30, 2024

12440. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. Prove

$$\sum_{k=0}^{n-1} C_k = 2\sum_{k=1}^{n} \binom{2n}{n-k} \sin\left(\frac{(4k+1)\pi}{6}\right).$$

12441. *Proposed by Pakawut Jiradilok and Yuan Yao, Massachusetts Institute of Technology, Cambridge, MA.* A *divisibility chain* of length n is a list $(a_1, \ldots, a_n)$ of positive integers such that either $a_{i+1}/a_i \in \mathbb{Z}$ for all $i \in \{1, \ldots, n-1\}$ or $a_i/a_{i+1} \in \mathbb{Z}$ for all $i \in \{1, \ldots, n-1\}$. For each positive integer n, determine the smallest positive integer N such that for any permutation π of $\{1, \ldots, N\}$, there exist indices $t_1 < \cdots < t_n$ such that $(\pi(t_1), \ldots, \pi(t_n))$ is a divisibility chain.

12442. *Proposed by Luu Dong, Hanoi National University of Education, Hanoi, Vietnam.* In $\triangle ABC$, let D, E, and F be the points at which the incircle of $\triangle ABC$ touches the sides BC, CA, and AB, respectively. Let P and Q be the points on line EF such that $\angle PBA = \angle QCA = \pi/2$. Prove that the common chord of the circumcircles of $\triangle PEC$ and $\triangle QFB$ passes through D.



12443. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Let n be an odd positive integer. Evaluate

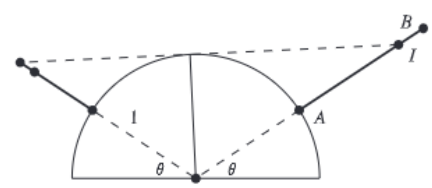
$$\sum_{j=1}^{n-1} \sin\left(\frac{j^2\pi}{n}\right) \cot\left(\frac{j\pi}{n}\right).$$

12444. *Proposed by Tho Nguyen Xuan, Hanoi University of Science and Technology, Hanoi, Vietnam.* Let p be an odd prime, and let a be a positive integer. Prove that the equation $m^2 + 1 = ap^n$ has at most two solutions in positive integers m and n.

12445. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.* Let M_n be the set of n-by-n complex matrices. We write I for the identity matrix in M_n. Suppose that $A, B \in M_n$ satisfy $(I - A)^2 B + 2(I - A)B(I - A) + B(I - A)^2 = B$, and suppose that there are nonzero complex numbers α, β and odd positive integers j, k such that $A^j = \alpha I$ and $B^k = \beta I$. Prove that the following two conditions are equivalent:
(1) There exists a finite subset S of M_n containing B such that $AX + XA \in S$ for all $X \in S$; and
(2) the matrix A is $(1/2)I$.

12446. *Proposed by Rolando Seclen, University of Piura, Lima, Peru, and Michael Elgersma, Esparto, CA.* Imagine a planet that is a perfect sphere of radius 1, and model a person as a segment AB with A on the surface of the planet and with B chosen above the planet so that AB is perpendicular to the surface. The length of AB is the *height* of the person. Each person has an eye I located on AB so that AI/AB is equal to a universal constant c with $0 < c < 1$.

Two people of heights h_1 and h_2 with $h_1 < h_2$ stand at antipodal points on the equator of the planet. They start walking simultaneously and at the same speed directly toward the north pole. We say that one of these people can see the other when the line connecting the eye of the first person to the top of the second person does not intersect the planet. The diagram shows the case where the taller person first sees the shorter.



(a) For which values of h_1 is it true that, for any c, there is an h_2 such that the taller person first sees the smaller person before the smaller person first sees the taller?

(b) For which latitudes θ are there c, h_1, and h_2 such that each person first sees the other when at latitude θ?

Problems from Mathematics Magazine, due July 1, 2024

2186. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Evaluate

$$\int_0^1 \frac{\operatorname{arctanh}\left(x\sqrt{2-x^2}\right)}{x}\,dx.$$

2187. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.*

For $r > s \geq 0$, evaluate

$$\prod_{n=0}^{\infty}\left(1+\frac{\cosh 2^n s}{\cosh 2^n r}\right).$$

2188. *Proposed by Kent Holing, Trondheim, Norway.*

Three tangent lines to a parabola form a triangle. Show that the circumcircle of the triangle passes through the focus of the parabola.

2189. *Proposed by Sela Fried, Israel Academic College, Ramat Gan, Israel.*

Let n be a positive integer and let $0 \leq k \leq n$. How many permutations $(x_1, \ldots, x_{2n})$ of $\{1, 2, \ldots, 2n\}$ satisfy

$$|\{1 \leq i \leq n \;:\; x_i > x_{i+n}\}| = k?$$

2190. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

Recall that the dihedral group is defined in terms of generators and relations as

$$D_{2n} = \left\langle r, s \mid r^n = 1, s^2 = 1, srs^{-1} = r^{-1}\right\rangle.$$

What is the probability that two (not necessarily distinct) elements of D_{2n} commute?

Problems from the College Mathematics Journal, due July 15, 2024

1266. *Proposed by Jeffrey Stuart, Pacific Lutheran University, Tacoma, WA.*

Let n be a positive integer. Now define the matrix A by $a_{jk} = j^2 + k^2$ whenever $1 \le j, k \le n$.

1. Determine the rank of A as a function of n.
2. When $\text{rank}(A) < n$, show that there is a basis for the nullspace of A consisting of vectors such that the first entry of each vector is positive, the second entry of each vector is negative, and the remaining nonzero entry of each vector is 1.

1267. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.*

A triangle with vertices A, B, and C is inscribed in a cardioid in such a way that side AB of the triangle intersects the cardioid in three points. Prove that the inscribed triangle has area $\frac{2}{3\pi}(1 - \cos\gamma)(\sin(\gamma - \varphi))$, where φ is the angle between the positive x-axis and the ray $\overrightarrow{OB}$, and γ is the angle between the positive x-axis and the ray $\overrightarrow{OC}$, where O is the cusp of the cardioid.

1268. *Proposed by Joseph Santmyer, United States Federal Government (retired).*

Let a and b be distinct complex numbers, and let m and n be positive integers. Prove that for any $z \in \mathbb{C}$,

$$(a-b)^{m+n-1} = (z-b)^n \sum_{k=0}^{m-1} (-1)^k \binom{n-1+k}{k} (z-a)^k (a-b)^{m-1-k} + (-1)^m (z-a)^m \sum_{k=0}^{n-1} \binom{m-1+k}{k} (z-b)^k (a-b)^{n-1-k}.$$

1269. *Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring with identity $1 \neq 0$. For the purposes of this problem, define the *non-unit covering number* of R to be the smallest cardinal number κ such that the set of non-units of R can be expressed as the union of κ-many ideals. Prove or disprove the following:

1. If either R has but finitely many maximal ideals or R has a finite non-unit covering number, then the non-unit covering number of R is the same as the cardinality of the collection of maximal ideals of R.
2. If either R has countably many maximal ideals or R has covering number at most $\aleph_0$, then the non-unit covering number of R is the same as the cardinality of the collection of maximal ideals of R.

1270. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Calculate the following integral (with full justification and without appeal to numerical software):

$$\int_0^\infty \frac{(3x^3 + 7x^2)\,dx}{(x^3 + x + 1)(x^3 + 4x + 8)}.$$

Problems from the American Mathematical Monthly, due July 31, 2024

12447. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* A *dissection* of a triangle T is a finite set of interior-disjoint triangles whose union is T. Determine all triples of positive real numbers (α, β, γ) with $\alpha \geq \beta \geq \gamma$ for which a triangle with angles α, β, and γ admits a dissection into triangles all of whose angles are smaller than α.

12448. *Proposed by Leonard Giugiuc, Drobeta–Turnu Severin, Romania.* Given real numbers $a_1 \ldots, a_n$, let $S_1 = \sum_{i=1}^n a_i$ and $S_2 = \sum_{i=1}^n a_i^2$. Prove

$$S_1\sqrt{nS_2} \leq S_1^2 + \frac{1}{2}\left\lfloor \frac{n^2}{4} \right\rfloor \left(\max_i a_i - \min_i a_i \right)^2 .$$

12449. *Proposed by Veselin Jungić, Simon Fraser University, Burnaby, Canada.* Let n be a positive integer with $n \geq 2$. The squares of an (n^2+n-1)-by-(n^2+n-1) grid are colored with up to n colors. Prove that there exist two rows and two columns whose four squares of intersection have the same color.

12450. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden.* Let n be an odd positive integer, and suppose that $x_1, \ldots, x_n$ are chosen randomly and uniformly from the interval [0, 1]. For $1 \leq i \leq n$, let $y_i = x_i - x_i^2$. What is the expected value of the median of $\{y_1, \ldots, y_n\}$?

12451. *Proposed by Adam L. Bruce, Dexter, MI.* Let A and B be complex n-by-n and n-by-m matrices, respectively, let $0_{m,n}$ denote the m-by-n zero matrix, let I_m denote the m-by-m identity matrix, and let exp be the matrix exponential function. Prove

$$\exp\begin{bmatrix} A & B \\ 0_{m,n} & 0_{m,m} \end{bmatrix} = \begin{bmatrix} \exp(A) & \left(\int_0^1 \exp(tA)dt\right)\cdot B \\ 0_{m,n} & I_m \end{bmatrix}.$$

12452. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* A *symmedian line* in a triangle is the reflection of a median across the corresponding angle bisector. The *symmedian point* of a triangle is the intersection of the three symmedian lines. Let $\triangle ABC$ be a nonequilateral triangle with circumcircle ω, and let a, b, c be the lengths of sides BC, CA, AB, respectively. Suppose P is a point on ω that minimizes or maximizes $a^4PA^4 + b^4PB^4 + c^4PC^4$ over all points on ω. Prove that P, the center of ω, and the symmedian point of $\triangle ABC$ are collinear.

12453. *Proposed by Askar Dzhumadil'daev, Almaty, Kazakhstan.* A *derangement* is a permutation with no fixed points. Let D_n be the set of derangements on $\{1, \ldots, n\}$. For $\sigma \in D_n$, let

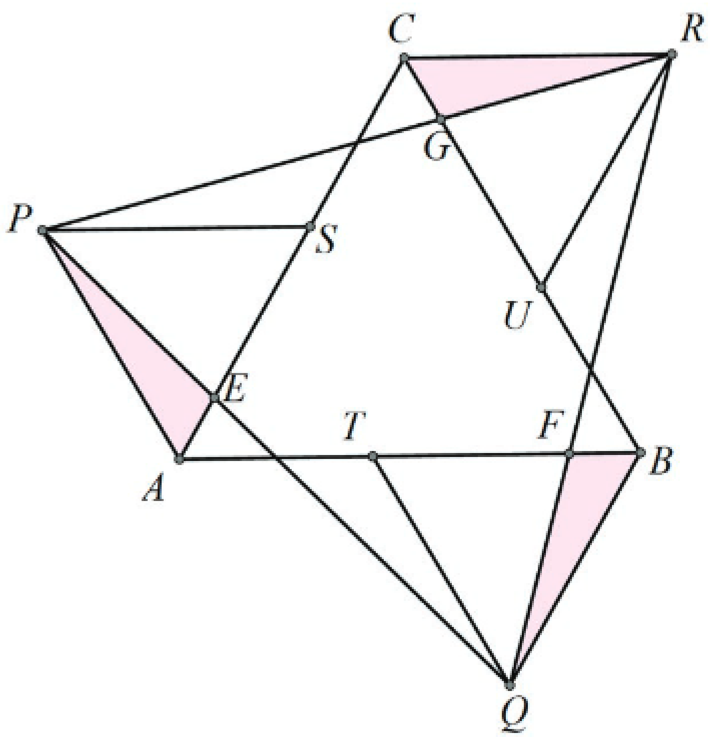
$$\omega(\sigma) = \frac{1}{\prod_{i=1}^n (\sigma(i) - i)} \qquad \text{and} \qquad \lambda_n = (-1)^{\lfloor n/2 \rfloor} \sum_{\sigma \in D_n} \omega(\sigma).$$

Prove $\lambda_n = 0$ when n is odd, and $\lambda_n = q_n^2$, where q_n is rational and $q_n > 1$, when n is even.

Problems from Math Horizons, due July 31, 2024

Equilateral Triangles (P466). Arsalan Wares (Valdosta State University) proposed this problem. Consider the equilateral triangle ABC with unit side length depicted in figure 1. Points S, T, and U are on the sides of ABC such that the triangles APS, BQT, and CRU are congruent equilateral triangles. Moreover, let $E = PQ \cap AC$, $F = RQ \cap AB$, and $G = PR \cap BC$. Suppose $CS : SA = 1 : r$ where r is a positive real number. Find the value of r that will maximize the sum of the areas of triangles AEP, BFQ, and CGR.

Figure 1. Configuration of equilateral triangles.



Activity Scheduling (P467). This issue features a second Sandbox problem offered by Jacob Siehler (Gustavus Adolphus College). Suppose a mathematician decides to schedule a certain activity according to the following rules. Each day, 1) if the activity has not been done in the past three days, then it must be done today; 2) if the activity was done yesterday, then it will not be done today; and 3) otherwise, randomly decide whether or not to do the activity today by flipping a fair coin. In the long run, what fraction of days will mathematicians be doing the activity?

Mindcraft Treasure Trove (P468). Michael Weselcouch (Roanoke College) suggested this *Mindcraft* version of a classic problem connected with his article "Math and Mindcraft?" (see page 10). In *Mindcraft*, single chests and double chests occupy one- and two-unit squares, respectively, and double chests can be placed horizontally or vertically. Find a recursive formula for the number of ways to arrange single and double chests to fill a $2 \times n$ array.

Sum of Cubes Inequality (P469). Danesh Forouhari (San Francisco Bay Area) devised this problem. Let $a, b, c \geq 0$. Prove that

$$a^3 + b^3 + c^3 \geq ab\sqrt{ab} + ac\sqrt{ac} + bc\sqrt{bc}.$$